Interpreting combinatorial problems as multiplicative structures

Frode Rønning

Norwegian University of Science and Technology, Norway; frode.ronning@ntnu.no

This paper is a study of two sessions from a Norwegian classroom where pupils in grade 4 (9 year olds) work on two combinatorial problems. The sessions are part of the activity in a research and development project where researchers from the university work together with teachers in two primary schools. The sessions described in this paper take place at one of the schools and my main interest is to study how differences between the situations described in the combinatorial problems may influence the children's strategies in their attempts to solve the problems. Although both situations are situations that the children are familiar with, and they can be seen as mathematically equivalent it turns out that they are handled in quite different ways. Using theory of multiplicative structures I will try to explain why the two situations turn out so differently.

Keywords; Elementary school mathematics, multiplication, combinatorial problems

Introduction

This paper is based on the research and development project LaUDiM, "Language Use and Development in the Mathematics Classroom, a collaborative project between the university and two primary schools. The project has special emphasis on language development, in particular to improve pupils' proficiency in expressing mathematical concepts and ideas using a variety of representations and to improve their proficiency in mathematical reasoning and justification. This paper is based on classroom sessions where children (age 9) work on combinatorial problems.

Combinatorial problems are well suited for developing mathematical reasoning and justification because the problems are easy to formulate and to attack. Yet, there are no clear algorithms to be used and it is also not obvious when the task is completed, i.e. when all possible combinations have been found. This creates a need for reasoning. Compared to other areas of whole number arithmetic there is relatively little research in this area. Already Piaget was interested in these problems (e.g. Piaget & Inhelder, 1975) and later they have also been addressed by others (English, 1991; Maher & Yankelwitz, 2011; Shin & Steffe, 2009) but still the literature on this field is not very rich.

Being able to see combinatorial problems as multiplicative structures would be helpful to develop strategies for solving them that go beyond mere random trying out combinations. My main interest in this paper is therefore to investigate to what extent the children see the situations as multiplicative and also to see what kind of semiotic representations the children invent.

My main theoretical framework for explaining the observations is the theory of multiplicative structures as laid out by Vergnaud (1983). Using a qualitative approach, the analysis is based on video recordings from the classroom as well as written material produced by the pupils.

Theoretical framework

Combinatorial problems represent a special kind of multiplicative structure. According to Steffe, "[f]or a situation to be established as multiplicative, it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit" (1994, p. 19). One of these composite units is used as a counting unit and one challenge is to identify that composite unit which is to be used as the counting unit.

Vergnaud (1983, p. 128) presents three classes of multiplicative structures – *isomorphy of measures*, *product of measures*, and *multiple proportions*. The class *product of measures* can be seen as a mapping from a product of two measure spaces into a third measure space, i.e. $M_1 \times M_2 \to M_3$. This class is essentially different from the other classes in that it invokes a measure space, M_3, which is not present in the initial situation. Combinatorial problems are typical representatives of a *product of measures* situation. Such situations are *symmetric problem situations* (Verschaffel & De Corte, 1996) in the sense that the role of the numbers in the product can be freely interchanged. This can often not be done immediately in isomorphy of measures situations but only after a reinterpretation of the situation (Rønning, 2012). Seeing a situation within the class product of measures as a counting situation, counting is done neither in the measure space M_1 nor in M_2 but in the new space M_3.

Vergnaud puts "problems concerning area, volume, Cartesian product, work, and many other physical concepts" (1983, p. 134) in the category product of measures. I will claim that even though both area (or volume) and combinatorial problems can be seen as product of measures, there is a significant difference between them. About area Vergnaud writes that "[t]he units of the product are expressed as products of elementary units", like "(1 unit of length) × (1 unit of length) = (1 unit of area)" (p. 134). However, it is perfectly reasonable to think of the area of a rectangle with length 4 cm and width 3 cm as an array with 4 cm^2 in each of 3 rows, so that the area is obtained by counting rows with 4 cm^2 in each, to obtain 12 cm^2. Then the situation can be seen as an isomorphy of measures situation. In combinatorial problems, however, the new unit is distinctly different from the original units so that if a counting procedure is to be applied, the counting unit is not present from the beginning. Also, the counting unit in combinatorics is a pair of units, not a singleton unit, and a challenge is that it is not clear from the beginning when to stop counting because the counting unit is of indefinite quantity (Shin & Steffe, 2009, pp. 170-171). A stopping strategy is described by Lyn English who introduced the term *odometer strategy*, which means that the child keeps the first item to be combined fixed and makes all possible combinations with this item. Then the first item is changed and all combinations with this possibility is exhausted, and so on (English, 1991, p. 460).

The tasks

The first task in the first session is about combining *shapes* and *colours*. The exact wording of the task is presented in Figure 1. Before the work started it was agreed that each biscuit should have icing in one and only one colour, all the shapes and all the colours should be used, and they have an unlimited amount of icing available. In the second session the task in Figure 2 was given, a task of the type often used in introductory combinatorics (see e.g. Maher & Yankelewitz, 2011).

How many different gingerbread biscuits can we make if we have cutters in these four shapes ☆ ◇ ○ ♡ and we have white, green and red icing?

Figure 1: Task 1

Ms. Hall has 3 pairs of trousers and 5 sweaters. The trousers are in the colours blue, black, and grey. The sweaters are in the colours blue, red, black, green and purple. She will use one pair of trousers and one sweater each day, and she will combine different pairs of trousers with different sweaters. How many days in a row can Ms. Hall wear different outfits?

Figure 2: Task 2

Both tasks represent combinatorial problems but there are some important differences between them. In the first task the two measure spaces M_1 and M_2 represent biscuits and colours, respectively, and M_3 represents coloured biscuits. However, coloured biscuits are still biscuits so in a counting situation one is still counting biscuits, just with an extra feature, the colour. In the second task M_1 and M_2 represent pairs of trousers and sweaters, respectively, whereas M_3 represents outfits, or days, as the task is formulated.

Method

The classroom episodes come from two separate sessions, on two different days in the same week. In a whole class situation on day 1, Task 1 is presented and conditions of the task are discussed. Then the pupils are put in pairs where they are supposed to find a solution to this task, as well as to other similar tasks. On day 2, Task 2 was worked with in a similar way but in different pairs than on day 1. The pupils show their workings on sheets of paper and they are free to use any representations and methods they might find appropriate. No indication is given that this may have something to do with multiplication and this can also not be inferred from what has been worked with in class in the immediate past. Selected pairs of pupils are video recorded, as well as all whole class situations. All written material from all the pupils is collected and is used as data. In this paper I follow two pairs (Naomi and Roger, and Nora and Lars) from the first session. In the second session I follow Naomi and Filipa, and Nora and Roger.

The analysis is based on principles from ethnomethodological conversation analysis, focusing on the thematic development of an interaction rather than on its structural development (Holstein & Gubrium, 2005). The episodes are analysed utterance by utterance to grasp the naturally occurring social interaction between the children in order to discern statements that may be interpreted as expressing a multiplicative situation. The workings on the children's worksheets are used to support the conversation analysis in two ways; the completed worksheets are available as data, and the process of creating the worksheets can be observed on the video along with the conversation taking place in this process, thereby providing additional information about the conversation. The semiotic representations emerging on the worksheets are analysed with respect to possible representations of multiplicative structures.

Analysis

Task 1

Naomi and Roger start looking at the task and Roger's first suggestion is that there will be seven different possibilities since there are four shapes and three colours, hence additive thinking. Then Naomi starts drawing the four shapes in one row and she colours the heart red. She indicates that she can continue to draw new rows with the same shapes, and change the colour for each row. She does not complete the drawing in detail but on the video it can be seen that she indicates three rows with four figures in each row. Then she counts, one-two-three, four-five-six, seven-eight-nine, ten-eleven-twelve, pointing to the drawing as she counts. It seems that Naomi has identified a countable unit (coloured shapes) that she counts in threes. She now considers herself finished with the task and Roger does not object. I observe that the first task was solved quickly, so I challenge them to find out what would happen if they had eight shapes and seven colours. They cannot really think of eight different shapes so Naomi just draws eight circles in a row and imagines that these circles represent eight different shapes. She then fills in with more circles below. They start to colour each row in one colour (purple, blue, red, …), until they have used all seven colours and hence got seven rows. The result is shown in Figure 3. (They did not colour the first row, so they put in eight green dots at the bottom. The three shapes at the bottom of Figure 3 do not belong to this solution.)

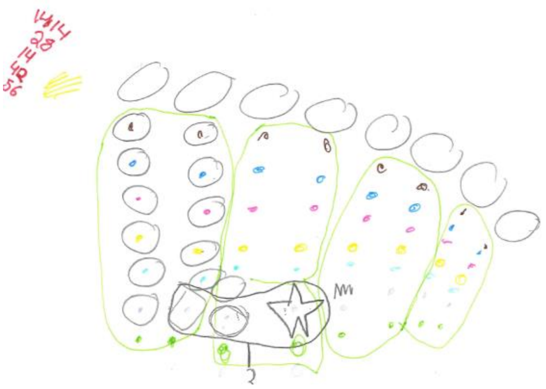

Figure 3: Naomi and Roger's solution

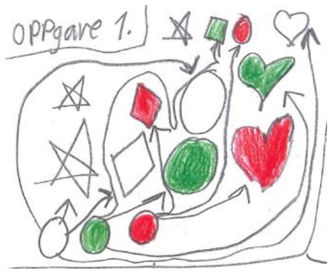

Figure 4: Nora and Lars' solution

In Figure 3 one can see a matrix structure with seven rows and eight columns and the pupils have grouped two and two columns together to ease the computation of the total number of objects. The calculation giving the result 56 can be seen in the top left corner of the picture.

Nora and Lars use a different representation where they draw the shapes on one line and make another line with three circles representing the colours. Then they draw arrows between the shapes and the colours but they also colour the shapes, and to cover all possibilities they draw new copies of the shapes and continue this process. Then they have represented all the 12 possibilities but not in a very systematic way. Lars says that "now each [shape] has got each colour one time". Nora says that the way Lars thinks is smart and that "we can make three rows because there are three colours". She then makes a new version of the drawing (Figure 4), with three rows but she does not keep each row in the same colour. In addition she draws arrows between the coloured circles and the shapes.

The matrix structure, used by both pairs, but most systematically by Naomi and Roger, shows a multiplicative structure; the colours are distributed over the biscuits (Steffe, 1994) in a way that gives equal groups of coloured biscuits. The counting scheme used by Naomi and Roger shows repeated addition and although multiplication is not explicitly mentioned by the pupils both the representations and the procedures point to a multiplicative structure.

Task 2

In Task 2 Naomi works together with Filipa, and Roger works together with Nora. Naomi and Filipa draw five sweaters and three trousers (see Figure 5) on the sheet and colour them according to the information in the task, except that instead of black they use brown. They start to connect sweaters and trousers with lines and on each line they write a weekday, starting with Monday. After some time it starts getting difficult to get the overview. When they have drawn nine lines (and marked nine days) they look at the drawing for about 20 seconds before they discover a new possibility and then after another 20 seconds they discover one more. The teacher, Ms Hall, comes in and asks if they have used all the sweaters and trousers. They say that they don't know and that the drawing gets a little messy. Soon they find three more possibilities so that they have 14. Naomi looks at the drawing of the brown sweater and says "All the trousers are on this one because there are three lines. I think we have made it. Nothing more is possible." Finally, Naomi finds one more solution and she writes "2 weeks and 1 day" and says: "We *think* it is two weeks and one day".

During this process the two girls have several times believed that they have found all the solutions but then they have found one more, and one more, until they have 15. However, they are not sure, which is reflected in Naomi's utterance, "We *think* it is two weeks and one day", with emphasis on "think", showing the challenge with not knowing where to stop (Shin & Steffe, 2009).

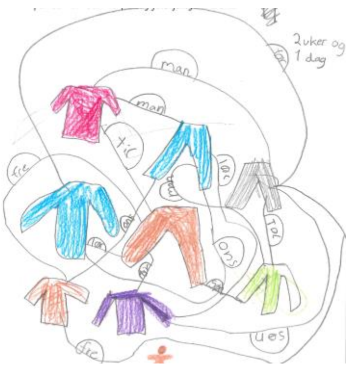

Figure 5: Naomi and Filipa's solution

Figure 6: Nora and Roger's solution

Their background for the conclusion is that they are not able to find more solutions but they have said so several times, so it might still be wrong. There is nothing in the situation, or in their choice of representation, that gives them feedback from which they can logically conclude that they have found all the solutions. They show fragments of reasoning when they start to count lines in the drawing. Naomi says that "there are three lines on every outfit" but after a prompt from one of the researchers she says "Yes, [three lines] on every sweater and every pair of trousers. No, there are five lines on the trousers. It is not easy to see if we have got all".

Nora and Roger choose a similar representation (Figure 6). They also start by drawing lines in a rather unsystematic manner and after having worked for quite some time Roger starts to reason on the situation (see dialogue below) together with one of the researchers.

Roger: We have been thinking that we can take all the sweaters with all the trousers and all the trousers with all the sweaters.

Researcher: Have you thought about how many it will be?

Roger: Fifteen. Five sweaters three times, that is fifteen.

Researcher: So you think it is 'times'.

Roger: Yes, because five times three is fifteen. And there are five sweaters and all five sweaters can be used three times. So then it is five times three.

Nora keeps working on the drawing and for every new combination she puts a tally mark in the top right corner (Figure 6). She asks Roger to count the tally marks, and he counts to 12.

Roger: OK, then twelve. So you were right (to Nora), it was twelve, not fifteen.

Nora: Yes, but … (she finds two more possibilities.)

Roger: No, perhaps we have forgot more. Yes, this one (points to the black sweater) has only two. So then it is fifteen. I was right. YES!

Now Roger seems quite convinced that there are no more solutions. He had a conjecture about 15 but then he had doubts because at that point they only had 12 and they could not immediately find more. However, after a while, they found three more possibilities. Then Roger gets his first conjecture confirmed and he rephrases his reasoning about why the solution is given by three times five:

Roger: Because there are three pairs of trousers and then there are five sweaters. And all five sweaters can be used three times on each pair of trousers, no, one time on each pair of trousers.

Both Naomi and Filipa, as well as Roger and Nora, found the correct solution but whereas Naomi and Filipa are not quite sure that they have found all the possibilities Roger and Nora seem quite convinced. They have identified the multiplicative structure when saying “all the sweaters with all the trousers and all the trousers with all the sweaters”. Here they are distributing one composite unit over the other (Steffe, 1994) and they can do this both ways. Although they are most explicitly saying “five sweaters three times” and “all five sweaters can be used … one time on each pair of trousers”, indicating that the sweaters are distributed over the trousers, the utterance “all the sweaters with all the trousers and all the trousers with all the sweaters” indicates that they can also envisage distributing the trousers over the sweaters. Hence, they have realised that the structure is symmetric (Rønning, 2012) and they have a way to check the solution; it is “five times three”. Naomi and Filipa did not show that they identified a multiplicative structure, except for the statement “All the trousers are on this one because there are three lines.” They mainly relied on their empirical findings and concluded with 15 because they were not able to find more. They never said “three times five” or “five times three”.

Discussion

Mathematically the two tasks have the same structure (they are isomorphic) but even though Task 2 was worked with only two days after Task 1 the children did not see a connection between the two tasks. Studies reported on in (Maher, Powell, & Uptegrove, 2011) indicate that isomorphism is only recognised at a much higher age than the children in my study.

Using Vergnaud’s (1983) notation both tasks can be seen as examples of a mapping from a product of two measure spaces into a third measure space, $M_1 \times M_2 \to M_3$. In Task 1 M_3 contains “coloured biscuits”, so there is a direct connection between the measure spaces. An element in M_3

can be represented as a composition of elements from M_1 and M_2. More precisely, an element $m_3 \in M_3$ can be seen as $m_3 = m_1 \times m_2$, where $m_1 \in M_1$ and $m_2 \in M_2$. This is similar to the area model, where the unit cm^2 can be seen as cm $\times$ cm. Similar to the area model one can obtain the result by counting with equal groups, counting "coloured biscuits". This is what most of the children did, and sometimes in a very systematic way, as shown in Figure 3. In this case it is possible to represent the elements of M_3 using the elements of M_1, biscuits, with an additional feature, colour, given by M_2. The grouping and counting process takes place on biscuits.

In Task 2 the elements of M_3 are not just elements of M_1 or M_2 with an additional feature. The composite unit is not a pair of units from the initial situation; it is a new unit, "outfit". Also, since the task asks for an answer in number of days one can see M_3 as consisting of days, or one can see a one-to-one mapping from M_3 consisting of outfits to an isomorphic measure space M_4 consisting of days. Many children chose a representation by lines, as seen in Figures 5 and 6 and to find the solution they counted lines. In this process they easily lost track of what they had done because often they were not systematic. One can see attempts of trying to be more systematic, for example by looking for all the possibilities with one fixed sweater. This is what English (1991) refers to as the odometer strategy.

As explained above I see the situation in Task 2 as conceptually different from the situation in Task 1. The unit "day" is not composed of the original units. Therefore I will call the situation in Task 2 a genuine product of measures model. The situation cannot be reduced to equal groups where counting can take place essentially on elements from one of the original measure spaces.

Also in Task 1, some pupils chose a representation with drawing lines between biscuits and colours (see Figure 4) but even here one can see a structure like a matrix. In the same session there were more tasks with biscuits and colours, just varying the numbers, and from the pupils' worksheets one can see that the matrix structure gets more visible in the second and third attempts. In Task 2 there is a greater variation in the choice of representations.

The size of the numbers involved in the two tasks is about the same (3×4 and 3×5) and all pupils found the correct solution in Task 1 whereas this was not the case in Task 2. Finding too few solutions can easily be explained because, as has been shown also in the dialogues, there is no indication where to stop. However, there is also one worksheet suggesting 20 combinations. Here one can see each pair of trousers drawn five times in each of the three colours. This would have been an adequate representation of the 15 outfits, each pair of trousers taken five times, i.e. one for each sweater. In addition the pupil has drawn the five sweaters, getting a total of 20.

Conclusion

In this paper I have presented two situations, both recognisable to the children in question. Both situations involve numbers in the same number range, and in themselves not representing any challenge for children of this age (9 years old). However, the choice of representation in the two situations turned out to be quite different and this I have explained by the difference in the relation between the input elements and the output elements in the two situations. In Task 1 the multiplicative structure comes more or less directly out of the representation, without much discussion among the children. In Task 2 the multiplicative structure is much more hidden and is revealed, if at all, only after long discussions. The two episodes show that combinatorial problems can be fruitful for enriching the topic of multiplicative structures but that potential for this depends on the specific situation. It could be argued that Task 1 was solved too easily to widen the children's scope on multiplicative thinking whereas Task 2 was more challenging. And when the situation in Task 2 was identified as a multiplicative, the multiplicative structure was expressed much more explicitly than in Task 1.

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