Conceptions and skills of fifth graders related to algebraic equation solving in three pilot tests

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Algebra is a content domain in which Finnish pupils have the most problems achieving good results in international comparisons. In a development and research project on early algebra, the authors are designing teaching materials and approaches to help primary school pupils construct important meanings and ideas connected to algebraic equation solving while still studying arithmetic. In order to get a better picture of the state and development of the algebraic conceptions and skills of the Finnish primary school pupils, the authors are developing tests for large numbers of first, third and fifth graders. In this paper, the authors report on the fifth graders' conceptions and skills related to algebraic equation solving in three pilot tests, which were conducted to inform the construction of the final test. Although a reasonable proportion of the pupils showed signs of different types of algebraic thinking, they only demonstrated an elementary understanding of the concept of a variable.

Keywords: Algebra, early algebra, equations, equal sign, variables.

Introduction

Finnish pupils have performed well in international comparisons of achievement in mathematics, especially when mathematical literacy has been the focus of the testing. However, when knowledge and skills in more traditional content areas are measured, the picture is less positive. For example, in TIMSS 2011 research, Finnish eighth graders performed below the international mean on algebra-related questions (Mullis, Martin, Foy, & Arora, 2012).

Algebra can be seen as a gatekeeper to more advanced mathematics and full participation in many fields in society. The transition from arithmetic to algebra causes problems for a great number of pupils and teachers every year. Early algebra is a field in mathematics education that seeks to develop thinking and communication processes that prepare children, from the early grades, for encountering algebra in their later studies. The proper use of traditional symbolism is not necessary for early algebra. Rather, the focus is on developing mental habits, where generalization and mathematics as a science of structures lie at the heart and mathematical communication is utilized in sense making. (Kieran, Pang, Schifter, & Fong Ng, 2016.)

As part of the national LUMA Finland development project funded by the Ministry of Education and Culture, the authors are developing early algebra teaching materials for grades 1–6, in line with the new National Core Curriculum (Finnish National Agency of Education, 2014). In Finland, algebra can be seen as the part of mathematics which begins with solving equations. However, equation solving in the primary grades usually involves so-called arithmetic equations. Finnish pupils do not encounter algebraic equations before grades 7 and 8, the solutions to which are based on very different conceptual foundations.

In our development and research project, we first want to understand Finnish primary school pupils' current algebraic thinking related to equation solving. Later, we want to see whether the implementation of our materials has had a large-scale impact. Thus, we are planning to test large groups of first, third and fifth graders in Rovaniemi, Finland in autumn 2018, and we will follow the same pupils two years later at the end of spring 2020. We are currently developing a test for the fifth graders. We have tried different items in three pilot tests in four different schools. The aim of this paper is to study the kinds of algebraic meanings related to equation solving that can be interpreted in the students' responses to the questions in the pilot tests. This information will help us to choose items for the final test.

Theory

In this research project, we employ a social constructivist view of learning (Cobb & Yackel, 1996). Individuals construct meaning from both their experiences and the social interactions in which they participate (Bauersfeld, 1988). The mathematical learning processes of individuals result in *mathematical conceptions*. These conceptions play a role in the formation of the social processes of mathematics classrooms when people act together and interpret each other's actions. They also play a role when pupils answer the questions in tests. Interpretations can be made about the students' conceptions on the basis of their responses.

Learning to solve equations in a more formal way is problematic for many pupils in Finnish lower secondary mathematics classrooms. During their primary school years (grades 1–6), the pupils have mostly solved so-called arithmetic equations (Filloy & Rojano, 1989), where the unknown appears on one side of the equation while there is a number on the other. The operational meaning (Herscovics & Linchevski, 1994; Kieran, 1981) of the equal sign as a "do something signal," known number facts, and inverse operations are sufficient for making sense of and solving such equations. For example, $x + 5 = 8$, $13x = 39$, and $6(x + 3) = 48$ are arithmetic equations.

When the variable in the equation appears on both sides of the equal sign (e.g., $4x + 2 = 7 + 2x$), arithmetic understanding is no longer enough. This is also true for equations where the solutions are negative (e.g. $2 – x = 7$) or when there are several occurrences of the variable (e.g., $6x + 5 – 7x = 27$). When solving these *algebraic equations*, one must acknowledge the relational meaning (Herscovics & Linchevski, 1994; Kieran, 1981) of the equal sign, which is "both sides have the same value," and also acknowledge that certain permitted actions on both sides preserve the balance and produce equivalent equations. In order to perform productive actions on both sides of the equation, the pupil needs to be able to view an expression as an object (Sfard & Linchevski, 1994), utilize inverse operations, and act on and with the variable (Herscovics & Linchevski, 1994). Herscovics and Linchevski (1994) emphasize that the demarcation between arithmetic and algebraic equation solving is found in the solution processes rather than in the mathematical form of the equations.

Connected to equation solving is the concept of the variable. A proper understanding of symbols as variables includes some key components. First, the symbol must be interpreted as representing "an unknown quantity," that is, a unit that does not have an ascertained value (Lucariello, Tine, & Ganley, 2014). Second, a pupil must interpret the symbol as representing a varying quantity or a range of unspecified values (Kieran, 1992). Third, a correct interpretation of variables entails awareness of a relationship between symbols, as they vary together in a systematic manner (Asquith, Stephens, Knuth, & Alibali, 2007; Kuchemann, 1978). Many pupils, however, hold erroneous conceptions about variables. Sometimes, pupils even ignore variables (Kuchemann, 1978). They may treat variables as a label for an object (Stacy & MacGregor, 1997) and believe that a variable is a specific unknown (Kuchemann, 1978). In Finnish elementary schools, pupils encounter the letter symbol only in the role of a specific unknown value in arithmetic equations. However, because variables are so important in algebra, Carraher and Schliemann (2007) recommend that mathematics educators should also treat unknown values in equations as indeterminates or variables. This means that the letter symbol is imagined to vary, and some of its values can make the equation true while others can make it false.

Methodology

In our overall research project, we use mixed methodology (Teddlie & Tashakkori, 2009). This paper is part of the quantitative line of the research. In line with constructivist epistemology (Lincoln & Guba, 2011), we maintain that the knowledge produced in the project, either qualitatively or quantitatively, is constructed by the researchers.

This paper reports on a part of the process of constructing a test for fifth graders to study the emergent algebraic conceptions connected to the equation solving of Finnish primary school pupils. We first searched the literature to establish what important meanings and skills, in general, form the basis for the pupils' understanding of algebraic equation solving. We then selected from the literature tasks that seemed suitable for our purposes, and we also constructed some tasks independently. To allow us to try a sufficient number of items, we constructed three 45-minute pilot tests. Tests A, B, and C included 11, 17, and 12 items, respectively. Test A was administered to 59 fifth graders (11 to 12 years old) in three classes in one school. Test B was taken by a total of 58 pupils in three classes in two different schools. The third test, Test C, included some revised items from tests A and B in addition to new questions. It was taken by 36 fifth graders in two classes in one school. The schools are located in Rovaniemi, Finland, and one medium-size municipality in Lapland.

In each pilot test, we encouraged the pupils to provide answers according to their own thinking and not to be influenced by other people's expectations, for example, the teacher's. The pupils were also informed about the purpose of the testing: to examine the test items. The teachers reported that there was enough time for the pupils to complete the test. The data for this research consist of the pupils' responses to the questions in the three pilot tests. We classified the pupils' answers to each question in every test. We didn't use theoretical constructs in the classification because we were interested in the pupils' genuine thinking and variation in it. Often the categories were evident from a teacher's perspective, such as when three different numbers were provided as a response or when there was evidence of cancelling an operation rather than calculating stepwise. Finally, we compared our findings to the theory.

We are aware of the problems in trying to understand pupils' thinking solely on the basis of written tests. Our interpretations are based on our knowledge of theory and our experience as teachers and teacher educators. We are both qualified mathematics teachers currently working in teacher training in the University of Lapland. We follow mathematics lessons in our teacher-training school, which is a primary school, continuously throughout the year.

We consider the following research question: What kinds of algebraic conceptions related to equation solving can be interpreted from the pupils' responses to the items in the three pilot tests?

After answering the research question, we can make judgements about the suitability of the items for the final test for fifth graders.

Results

In the following sections we report on and discuss the pupils' answers to the test items that included at least a few responses demonstrating the presence of algebraic meanings. The interpretation of answers to the items that did not elicit algebraic meanings is beyond the scope of this paper.

Meanings attributed to the equal sign

We found three questions to be informative for interpreting the meanings the pupils gave to the equal sign. First, at the beginning of Test A, we presented a missing number task to the pupils: 8 + 4 = __ + 5. Sixty-one per cent of the pupils provided number 7 as the answer, thus demonstrating their understanding of the relational meaning of the equal sign. Almost a third of the pupils (29 %) answered 12 or 17, which conveys that they viewed the equal sign as a signal for giving the answer. They acted according to the operational meaning of the sign. The percentage of correct answers in Test A was considerably higher than in the research conducted by Falkner, Levi, and Carpenter (1999), who found that only 10 % of all the pupils in grades 1 to 6 in their study provided the answer 7 to this question. Recent research has shown that around 30 % of American fourth and fifth graders can solve equations with operations on both sides of the equal sign correctly (Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). It should be noted that the students in test A were posed the same equation a year earlier in an informal test performed by the authors. Forty-two per cent of the pupils provided the number 7 as an answer at that time.

Second, in Item 1, Test C, we asked the pupils to give six different decompositions for the number four in the following form: 4 = __ + __ = __ + __ = …. Forty-two per cent of the pupils provided at least a few different decompositions, using the equal sign correctly according to the relational meaning. However, 39 % of the pupils wrote a chain of equations, for example, 4 = 2 + 2 = 4 + 4 = 8 + 2 = 10 + 1 = 11 + 2 = 13 + 2 = 15. The latter type of answer is comprehensible only by understanding that those pupils attributed the operational meaning to the equal sign in this situation.

Third, in Item 5, Test B, we posed a direct question to the pupils: "What does the equal sign '='mean in your opinion? Can it mean something else?" Twenty-seven per cent of the pupils provided responses such as "Sign '=' means in my opinion the same amount or equal to," conveying a conception of the "relational meaning" as their first choice. Two pupils (3.4 %) who provided the "relational meaning" as their first choice also indicated that the equal sign could mean "after it becomes the answer." The "operational meaning" was provided as the first choice by 31 %

of the pupils. Six of them (10 %) also reported the "relational meaning." Thirty-one per cent of the pupils in Test B provided no answer or an irrelevant one. Three pupils (5.2 %) simply wrote down the sign as it is pronounced in Finnish.

Finally, we also tested whether the pupils taking Test C were able to demonstrate an understanding of the symmetry and transitivity of the equal sign in the context of algebraic expressions. In Question 10, we asked the following: "If a = 2 + x and 2 + x = 15, what is the value of a then?" Eight pupils (22 %) justified the answer 15 by referring to the transitivity property, for example, "15 because it says how much 2 + x is," and four (11 %) provided the right answer without explanation. Five pupils (14 %) had clearly solved x = 13 first and then found the value of a by adding two. Thirty-nine per cent of the pupils did not answer the question. In Question 11 we asked the following: "It is known that 2x + 4 = 3y. Is the following true? 3y = 2x + 4?" Twenty-five per cent of the pupils provided a justified acceptable answer "yes" while 14 % only gave the correct answer. Three pupils (8.3 %) substituted concrete values for the variables, and 42 % of the pupils failed to give a mathematical answer.

Interpreting the letter symbol

Firstly, in Test C, we asked the following question: "Your class is making a trip. You visit a supermarket where everyone buys some pick 'n' mix candies. The bag weighs 20 g. If the weight shows number Y g, how can you write the weight of the sweets?" Two (5.5 %) pupils answered Y – 20 g, demonstrating some confidence in using the letter symbol for the weight. Sixteen per cent of the pupils gave a particular value for Y and subtracted 20 g. Sixty-six per cent of the pupils in Test C failed to provide any mathematical answer to the question. According to Stacey and MacGregor (1997), 50 % of Australian pupils of approximately the same age solve similar problems correctly in the beginning of their algebra studies.

Secondly, we also asked explicitly about the meaning of the variable in the expression $2 \cdot x + 1$ in Test A. Fourteen per cent of the pupils answered that x was "joku luku" (in Finnish), for which the translation could be "some number" or "a number." It can be interpreted that those pupils were only referring to the nature of the letter symbol. We maintain that this is an appropriate basic phase (Lucariello et al., 2014) for pupils to start interpreting variables in algebra. Only three pupils in Test A (5 %) gave an answer similar to "as a matter of fact it could be any number." In algebra, the letter symbol must also be interpreted as representing a varying quantity or a range of unspecified values (Kieran, 1992). Fifty-three per cent of the pupils in Test A gave a particular value of x as an answer. One common misunderstanding of the concept of a variable is the conception that a variable is a specific unknown (Kuchemann, 1978; Stacy & MacGregor, 1997). Twenty-four per cent of the pupils replied that they did not know what to answer.

Third, the concept of a variable is related to the question of what numbers pupils can imagine the variable might be. In question 8 of Test A we asked the pupils to answer the following: "Ten was added to a number and the result was 7. Find the original number." Thirty-one per cent of the answers included negative numbers, 12 % of the pupils answered that "it is impossible to calculate," and 42 % failed to give any answer.

Finally, the last question in Test C was about the truth value of an equation for a particular value of the variable: “Is the equation 4x – 5 = x + 1 true, when x = 2?” In 22 % of the answers, there was an indication of substitution. Six pupils (17 %) performed the substitution correctly, and 53 % failed to answer the question.

Seeing expressions as objects

In Item 7, Test C, we presented the following question: “If V + U = 4, then find V + U + 6.” In 25 % of the answers, there was evidence of substituting “4” for the whole expression V + U in V + U + 6. Six pupils (17 %) gave an answer with no explanation. However, some of those six may have also perceived the expression V + U as a whole. Twenty-eight per cent of the pupils assigned particular values for V and U, for example, 1 and 3. Then, they calculated the value of the expression as 1 + 3 + 6 = 10. A further 25 % of the pupils failed to answer the question.

Using inverse operations

In Test C we asked about inverse operations in the context of pure mathematics. Question 2 was as follows: “Evaluate 17 + 59 – 59 + 18 – 18 =.” In 33 % of the answers, there was an indication of the “doing and undoing” strategy and of understanding addition and subtraction as inverse operations. Twenty-five per cent (25 %) of the pupils performed the operation correctly. Fifty per cent of the pupils chose the strategy of calculating the value of the expression step by step. However, only 17 % succeeded and arrived at the correct answer. This does not necessarily mean that those pupils were incapable of utilizing inverse operations. It may be that they were obedient to their teacher’s instructions for performing calculations. Twenty-three per cent of above-average Canadian seventh graders who were presented with this same question realized that no operations had to be performed because of cancellations (Herscovics & Linchevski, 1994). In Question 3, Test C, there was an expression involving multiplying and dividing by the same number. The cancelling strategy was used by 19 % of the pupils while 47 % used step-by-step calculations.

Acting on and with the variable

Test C also included Task 4: “a. If $2 \cdot x$ = y, what is $4 \cdot x$? b. If $3 \cdot a$ = b + 5, what is $6 \cdot a$?” In the first part, only two pupils (5.5 %) answered $2 \cdot y$. However, 9 pupils (25 %) either wrote that the result must be doubled or gave a symbolic answer such as yy or 2,y, which we interpreted as having the right meaning but unconventional syntax. At the time the tests were conducted, the pupils had not yet studied the use of symbols in algebra. Seven pupils (19 %) substituted x with a number, and 16 pupils (44 %) either did not provide an answer or wrote nonmathematical comments. The second part, Task 4, seemed to be more difficult for the pupils in Test C. One student (2.8 %) wrote that the result must be doubled and also gave the answer $B^2 + 10$. We interpreted this to mean that B^2 stood for b doubled. Five students (14 %) doubled either b or 10. Twenty-eight per cent of the students made a concrete substitution to get a value. Forty-seven per cent of the students in Test C failed to give a mathematical answer. All our items designed to test whether the pupils were able to act with the variable failed. They were equations that could be easily solved by substitution, and thus the need to group two unknown terms did not arise.

Discussion

Compared to American pupils (Falkner et al., 1999; Rittle-Johnson et al., 2011), the participants in the pilot tests seemed to more commonly attribute the relational meaning to the equal sign. Large groups of pupils in the tests also gave an operational meaning to the equal sign in mathematical tasks, which most likely prompted them to just "write down the solution after the equal sign." The same applies to the direct question about the meaning of the equal sign. A more substantial proportion of pupils should, however, master the relational meaning by the end of primary school. Research has shown that even the youngest pupils can understand this concept when properly instructed (Falkner et al., 1999).

Perhaps due to the tradition of teaching only arithmetic equations involving one unknown and not attending to the concept of variables in Finnish primary school mathematics, 53 % of the pupils in Test A had the common misconception that letter x in an expression means one specific number. Only three pupils (5 %) expressed the idea of a varying quantity (Kieran, 1992). In Test C, only two pupils (5 %) were able to write a simple expression including a variable, and two-thirds of the pupils failed to answer the question. However, there is evidence that second to fourth graders are capable of integrating algebraic conceptions of variables into their thinking (Carraher, Schliemann, Brizuela, & Earnes, 2006). This leads to a question: Is Finnish pupils' unfamiliarity with the concept of a variable one reason for their weak performance in algebra?

An interesting finding is that approximately one-fourth of the responses in many items in Test C demonstrated signs of the pupils' emergent algebraic thinking in the context of symbolic algebraic expressions. This applies to interpreting expressions as objects, acting on the variable (Herscovics & Linchevski, 1994), and applying the transitivity and symmetry of the equal sign. The students have not received any instruction in those areas.

In many questions, some pupils showed conceptions required in algebraic equation solving while others adhered to working with concrete numbers and previously taught arithmetic methods; still others failed to answer the questions at all. This reflects the heterogeneous character of Finnish classrooms. Items in the final test must be constructed in such a way that some fifth graders can answer them at the beginning of the school year and not all sixth graders can answer them at the end of the school year. Such a design could demonstrate the progress the pupils have made in two years of study. When trying to determine the meanings attributed to the equal sign, we must use the most difficult tasks. Other items, especially those concerning the concept of a variable, seem to allow recognition of possible progress. We have a challenge in regard to those students who failed to answer many of the questions.

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